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Stochastic finite element method for random temperature in concrete structures

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Abstract

This paper proposes numerical methods for random temperature field in mass concrete structures to research various kinds of random variability including both random environmental temperature and random material properties. First presented, based on local average method of random field, are the formulas of random variational principle as well as the corresponding formulas of stochastic FEM. Then discussed is the stochastic spectral response finite element method, given the complex frequency functions as random functions of material parameters, to resolve the effect of random environmental temperature. Finally, the Taylor expansion series is referred to simplify the non-stationary process and suggest the corresponding formulas for the effect of random hydration heat of concrete. The paper demonstrates that these proposed methods can help to study conveniently the random variability of temperature, which could appear as both random processes and random fields in mass concrete structures. Numerical examples calculated by the methods, theoretical and Monte-Carlo simulation are also provided. © 2001 Published by Elsevier Science Ltd.

Keywords: Random temperature field; Random variational principle; Stochastic finite element method; Mass concrete structures

1. Introduction

Thermal stress can sometimes act as a principal factor for the failure of concrete structures. Accordingly, it is important to evaluate correctly the temperature of mass concrete structures during both the construction period and operation period. However, random variation of both environmental temperature and material properties has a significant influence on the temperature in mass concrete structures. It is necessary then to consider the temperature in mass concrete structure as a random temperature field.

The research was pioneered by Heller with his co-workers (Heller 1976a,b; Heller et al., 1979, 1983), who applied spectral method (the method of power response spectrum). This analytical solution is, however, inapplicable to concrete. Based on the Bessel and Kelvin functions, Tsubaki and Bazant (1982) suggested a

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Nomenclature

T	temperature
t	time
x, y, z	spatial coordinates in global coordinate system
x', y', z'	spatial coordinates in local coordinate system
$\mathbf{T}$	column matrix of temperature
a	temperature transfer coefficient
b	coefficient of surficial heat emission
λ	coefficient of thermal conductivity
θ	hydration heat of concrete
T_a	environmental temperature
n_a	number of different kinds of environmental temperature
n	number of scalar random fields
T_0	initial temperature
θ_0, m_1	parameters of concrete hydration heat
X	random variable
$S(\cdot)$	random field
A	fluctuation part of environmental temperature
$\mathbf{h}(\omega)$	column matrix of complex frequency function
N	shape function
n_{v}	number of random variables
N_p	number of nodes
$\mathrm{E}[\cdot]$	expectation
$\mathrm{var}[\cdot]$	variance
$\mathrm{cov}[\cdot]$	covariance
$\mathrm{R}[\cdot]$	auto-correlation function
ρ	correlation function of random field
V	element volume
ξ, η, ζ	local coordinates
H_k	Gaussian weighted coefficient
n_{g}	number of Gaussian points
$\lvert J\rvert$	Jacobi determinant
$\hat{\mathbf{T}}$	matrix of transformation
σ	standard deviation
l, m, n	orientation cosine

Subscripts

T_1, T_2	temperature solution of heat conduction equation-I and equation-II
ρ_{12}	correlation function in the oxy plane of the specific random field
ρ_3	correlation function along z' axis of the specific random field
ρ_{ij}	component of the correlation function matrix at the ith line and the jth column
$[\cdot]'_k$	differentiation with respect to the kth random variable
S_i	the ith scalar random field
$S_{\bar{V}i}$	local average in the element V_i
ik, jl	the kth Gaussian point of ith element and the lth Gaussian point of jth element
$([\cdot]_m)_l$	value at the lth Gaussian point with respect to the mth node
$[\cdot]_k$	value at the kth Gaussian point

Superscripts

$T^{(0)}$	temperature field of the heat conduction equation-I$^{(0)}$
$T^{(i)}$	temperature field of the heat conduction equation-I$^{(i)}$ $(i = 1, 2, \ldots, n_a)$
${}^{(i)}T$	temperature at the ith node
$[\cdot]^{\mathrm{T}}$	transpose
$[\cdot]^*$	conjugate
$\rho^{(i)}(\cdot)$	correlation function of the ith scalar random field
$\hat{\mathbf{T}}^{(i)}$	transformation matrix of the ith scalar random field

method of impulse response functions, which can be used for non-stationary aging system. However, it is computationally inefficient and sometimes the impulse response function can not be conveniently obtained. The generalized spectral method (the spectral finite element method) recommended by Bazant and his coworkers is based on the fact that the environmental effect can be adequately described by only a few sinusoidal components (Bazant, 1982, 1986; Bazant and Wang, 1984a,b,c; Bazant and Liu, 1985; Bazant and Xi, 1993). Evaluation of random temperature in computer structures has been done by Suzuki et al. (1988), Oktay and Kammerer (1982) and Shigekazu et al. (1993), among many others. However, by the suggested methods, neither random environmental factors nor the random heat source (in concrete structures is

hydration heat) has been taken into account. Therefore, these methods are not applicable to concrete structures.

Stochastic finite element method, a very efficient method to analyze random stress and deformation of large structures, has been developed for more than 10 years. There is a lot of literature, among which the study done by Hisada and Nakagiri (1982), Vanmarcke and Shinozuka (1986), Ghanem and Spanos (1991), Kleiber and Hien (1992), Zhu et al. (1992) should be mentioned. At present, however, it is known that, for large and complicated structures, the stochastic finite element method can only be implemented for two cases: one is that when material parameters are considered as random fields (or random variables), loads can not be simulated as random processes (random excitation). The other is that when loads are modeled as random processes, material parameters should be constant.

In order to analyze of random temperature field of mass concrete structures, material properties of concrete including temperature transfer coefficient and surficial heat emission coefficient should be modeled as random field or random variables. Meanwhile, the loads e.g., heat source or environmental temperature, should be modeled as random processes. For this reason, a new method for the stochastic finite element analysis needs to be employed.

The main purpose of this paper is to develop numerical methods of evaluating the random temperature field of mass concrete structures. The suggested methods are based on extended stochastic FEM, by which the effect of both random environmental temperature and random material properties can be conveniently taken into account.

2. Descriptions

According to the theory of heat conduction, temperature field $T(t,x,y,z)$ is the solution of the following equation:

$$\left.\begin{array}{l} \frac{\partial T}{\partial t} = a\nabla^2 T + \frac{\partial \theta}{\partial t} \\ T|_{t=0} = T_0(x,y,z) \\ \frac{\partial T}{\partial n}\big|_{\mathrm{c}} = -\frac{b}{\lambda}(T - T_a) \end{array}\right\} \tag{1}$$

in which a is temperature transfer coefficient; b is surficial heat emission coefficient; λ is coefficient of thermal conductivity; T_a is environmental temperature; T_0 is initial temperature of concrete; θ is hydration heat of concrete. This equation can also be expressed in the form of functional extreme value:

$$I(T) = \int\int\int_{\Omega} \left\{ \frac{1}{2}\nabla^2 T + \frac{1}{a}\left(\frac{\partial T}{\partial t} - \frac{\partial \theta}{\partial t}\right) T \right\} \mathrm{d}\Omega + \int\int_{\mathrm{c}} \left(\frac{b^*}{2} T^2 - b^* T_a T \right) \mathrm{d}s = \min \tag{2}$$

in which, $b^* = b/\lambda$.

As for mass concrete structures such as concrete gravity dams and arch dams, the environmental temperature T_a is the temperature of air surrounded or the temperature of water. Therefore, the environmental temperature can be generally expressed as follows:

$$T_a(t) = T_m + \sum_{i=1}^{n_a} T_{ai}(t) \tag{3}$$

in which, $T_a(t)$ is the environmental temperature; T_m is a part of $T_a(t)$ independent with time t; $T_{ai}(t)$ $(i = 1, \ldots, n_a)$ is a periodic stationary random process (sine or cosine process).

As for the temperature process of environmental air, we have:

$$T_a(t) = T_A \sin\omega(t - t_0) + T_B \tag{4}$$

where t_0 is the beginning time; T_A, T_B are random variables, the statistics of which can be obtained from observational data. Therefore, for the air temperature in Eq. (3), we have: $T_m = T_B$, $n_a = 1$. Moreover, as for the random temperature process of the water in reservoir, we have (Zhu et al., 1976)

$$\begin{aligned} &T_m = T_{wm}(z) \\ &na = 2 \\ &T_{a1}(t) = A_w(z)\cos\omega(t-t_0)\cos[\varepsilon(z)] \\ &T_{a2}(t) = -A_w(z)\sin\omega(t-t_0)\sin[\varepsilon(z)] \end{aligned} \tag{5}$$

in which, $T_{wm}(z)$ is the expectation of water temperature (°C) in the depth of z; $A_w(z)$ is the modulation function of the water temperature; $\omega = 2\pi/12$. In addition, $\varepsilon(z)$, a function of z, usually can be written as $\varepsilon(z) = c_1 + c_2\exp(-c_3 z)$, in which c_1, c_2 and c_3 can be regarded as random variables or just constants.

Hydration heat $\theta(t)$ is often expressed as follows (Zhu et al., 1976):

$$\theta(t) = \theta_0[1 - \exp(-m_1 t)] \tag{6}$$

in which, θ_0 and m_1 are random variables. Hence, in Eq. (1), we have

$$\frac{\partial\theta}{\partial t} = \theta_0 m_1 \exp(-m_1 t) \tag{7}$$

Obviously, $\partial\theta/\partial t$ is a non-stationary process. Under the influence of $\partial\theta/\partial t$, the system output (structural temperature field) is also a non-stationary process. Regarding large and complicated structures which have random material properties and need to be simulated by finite element method, there have been no reports on effective and efficient methods up to now for the case of external excitation, modeled as a non-stationary process. For this reason, this paper uses the first order Taylor expansion series to simplify the non-stationary process $\partial\theta/\partial t$ as follows:

$$\begin{aligned} &\frac{\partial\theta}{\partial t} = \frac{\partial\bar{\theta}}{\partial t} + \frac{\partial\Delta\theta}{\partial t} \\ &\frac{\partial\bar{\theta}}{\partial t} = \bar{\theta}_0\bar{m}_1\exp(-\bar{m}_1 t) \\ &\frac{\partial\Delta\theta}{\partial t} \approx \bar{m}_1\exp(-\bar{m}_1 t)\Delta\theta_0 + (\bar{\theta}_0 - \bar{\theta}_0\bar{m}_1 t)\exp(-\bar{m}_1 t)\Delta m_1 \end{aligned} \tag{8}$$

where $\bar{\theta}_0$ and $\bar{m}_1$ are the expectation of θ_0 and m_1, respectively.

$$\begin{aligned} &\Delta\theta_0 = \theta_0 - \bar{\theta}_0 \\ &\Delta m_1 = m_1 - \bar{m}_1 \end{aligned} \tag{9}$$

By using Eq. (8), the non-stationary process $\partial\theta/\partial t$ is simplified as $f_1(t) + f_2(t)\Delta\theta_0 + f_3(t)\Delta m_1$, where $f_1(t)$, $f_2(t)$ and $f_3(t)$ are definite functions of time.

The initial temperature T_0 and concrete material parameters a, λ and b can be simulated as random fields (Vanmarcke and Shinozuka, 1986; Zhu et al., 1992). If auto-correlation coefficient function of each random field is approximately equal to 1.0, or the auto-correlation coefficient function can not be obtained due to insufficient observational or test data, these parameters can be approximately regarded as random variables. Although the auto-correlation function of these parameters can not be easily obtained, for the integrity of this paper, the method of random field discretization in conjunction with the random variational principle is still applied and developed for random temperature field.

3. Random variational principle

According to Eq. (2), the first order functional variation of temperature can be obtained as follows:

$$\delta I(T) = \int\int\int_{\Omega} \delta T \left[a\nabla^2 T + \left(\frac{\partial \theta}{\partial t} - \frac{\partial T}{\partial t} \right) \right] \mathrm{d}\Omega - \int\int_{\mathrm{c}} \delta T \times a \left[\frac{\partial T}{\partial n} + b^*(T - T_a) \right] \mathrm{d}s = 0 \tag{10}$$

in which, $\partial T/\partial n$ denotes the differentiation of T with respect to the orientation cosine of the normal direction at the boundary of integration.

Let $X(x,y,z)$ denote a random field or a random variable, and let $S(X,x,y,z)$ denote a random function, which could be the random temperature function T, T_a as well as the random parameters θ, a, b and λ. The random function S can be expanded into the first order Taylor series at the point (x,y,z) and the expectation of the random field $\bar{X}(x,y,z)$ as:

$$S[X(x,y,z),x,y,z] \approx S[\bar{X}(x,y,z),x,y,z] + \left(\frac{\partial S[X,x,y,z]}{\partial X} \right)\Big|_{\bar{X}} \mathrm{d}X \tag{11}$$

In the above equation, it should be noticed that differentiation of the parameters a, b and λ with respect to themselves are 1.0, while the differentiation with respect to other parameters are zero. Let $\mathrm{d}X = (X - \bar{X})\varepsilon$, where ε is a very small positive number. Then the above equation can be written as the first order perturbation equation

$$S = \bar{S} + \varepsilon S' \tag{12}$$

in which, the prime denotes differentiation with respect to random variables.

Substituting the above equation into Eq. (10), we have

(a) The zero order variational principle

$$\delta^{(0)}(T) = \int\int\int_{\Omega} \delta T \left[\bar{a}\nabla^2 \bar{T} + \left(\frac{\partial \bar{\theta}}{\partial t} - \frac{\partial \bar{T}}{\partial t} \right) \right] \mathrm{d}\Omega - \int\int_{\mathrm{c}} \delta T \times \bar{a} \left[\frac{\partial \bar{T}}{\partial n} + \bar{b}^*(\bar{T} - \bar{T}_a) \right] \mathrm{d}s = 0 \tag{13}$$

(b) The first order variational principle

$$\begin{aligned} \delta I^{(1)}(T) = {} & \int\int\int_{\Omega} \delta T \left[(a'\nabla^2 \bar{T} + \bar{a}\nabla^2 T') + \left(\frac{\partial \theta'}{\partial t} - \frac{\partial T'}{\partial t} \right) \right] \mathrm{d}\Omega - \int\int_{\mathrm{c}} \delta T \times a' \left[\frac{\partial \bar{T}}{\partial n} + \bar{b}^*(\bar{T} - \bar{T}_a) \right] \mathrm{d}s \\ & - \int\int_{\mathrm{c}} \delta T \times \bar{a} \left[\frac{\partial T'}{\partial n} + \frac{b'\bar{\lambda} - \bar{b}\lambda'}{\bar{\lambda}^2}(\bar{T} - \bar{T}_a) + \frac{\bar{b}}{\bar{\lambda}}(T' - T'_a) \right] \mathrm{d}s = 0 \end{aligned} \tag{14}$$

As there is time t in Eqs. (13) and (14), the above variational principle can also be termed as the instantaneous variational principle for random temperature field, which can be used to deal with the influences of random parameters such as T, θ, a, b and λ. The influences of random environmental temperature and hydration heat of concrete being involved with time effect, can be handled by the methods discussed in the following sections.

4. Numerical studies

Up to now, many studies have been made on the random field discretization in conjunction with SFEM, such as those by Vanmarcke and Shinozuka (1986), Zhu et al. (1992). There are several methods for the discretization of random fields have been suggested, among which the local average method is readily

adopted in this paper for its simplicity, efficiency and its insensitivity to the correlation structures of random fields (Vanmarcke and Shinozuka 1986; Zhu et al. 1992).

When the structure is discretized into iso-parametric finite elements, suppose that random fields being considered are discretized into q parametric elements, according to the concept of local average of random field (Vanmarcke and Shinozuka, 1986), we have

$$X(x,y,z) \approx \sum_{i=1}^{q}\left[\varphi_i \sum_{k=1}^{n_g} X(\xi_k,\eta_k,\varsigma_k)|J|_k H_k \Big/ \sum_{k=1}^{n_g}|J|_k H_k\right] \tag{15}$$

in which, if $(x,y,z) \in \Omega_i$ (Ω_i as the domain of the ith element), $\varphi_i(x,y,z) = 1$, otherwise, $\varphi_i(x,y,z) = 0; n_g$ is the number of Gaussian point; H_k is the weighted coefficient for the kth Gaussian point; $|J|$ is the Jacobi determinant; ξ_k, η_k and ς_k are local coordinates of the kth element. From Eq. (15), correlation coefficient of random fields can be obtained. Specific formulas are shown in Appendix A. Let the random temperature field be discretized into N_e elements and N_p nodes, we have:

$$\begin{aligned} \bar{T} &= \sum_{i=1}^{N_p} N_i \bar{T}_i = \mathbf{N}\mathbf{T} \\ T' &= \sum_{i=1}^{N_p} N_i T'_i = \mathbf{N}\mathbf{T}' \end{aligned} \tag{16}$$

where $\mathbf{N}$ is global displacement shape function and $\mathbf{T}$ is global temperature column matrix. When the node being considered is in a specific element e, we have $N_i = N_i^e$, in which N_i^e is the shape function of the element e; otherwise, $N_i = 0$. Notice that $\mathbf{T}' = \sum_{i=1}^{q} \mathbf{T}'_k \Delta X'_k$, substituting Eq. (16) into Eqs. (13) and (14), stochastic element formulas for the random temperature can be obtained.

(a) The zero equation

$$\bar{\mathbf{K}}_\mathbf{1}\bar{\mathbf{T}} + \bar{\mathbf{K}}_\mathbf{2}\frac{\partial \bar{\mathbf{T}}}{\partial t} + \bar{\mathbf{F}} = \mathbf{0} \tag{17}$$

in which

$$\bar{\mathbf{K}}_\mathbf{1} = \sum_e \left[\bar{a}\int\int\int_{\Omega_e}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \bar{a}\int\int_{c_e}\frac{\bar{b}}{\bar{\lambda}}N_i N_j\,\mathrm{d}s\right] \tag{18}$$

$$\bar{\mathbf{K}}_\mathbf{2} = \sum_e \int\int\int_e N_i N_j\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{19}$$

$$\bar{\mathbf{F}} = \sum_e \left(-\int\int\int_{\Omega_e}\frac{\partial\bar{\theta}}{\partial t}N_i\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \bar{a}\int\int_{c_e}\frac{\bar{b}}{\bar{\lambda}}N_i\mathbf{N}\bar{\mathbf{T}}_\mathbf{a}\,\mathrm{d}s\right) \tag{20}$$

(b) The first order equation

$$\bar{\mathbf{K}}_\mathbf{1}\bar{\mathbf{T}}'_\mathbf{k} + \bar{\mathbf{K}}_\mathbf{2}\frac{\partial \mathbf{T}'_\mathbf{k}}{\partial t} + \mathbf{F}'_\mathbf{k} = 0 \quad (k = 1,2,\ldots,n_v) \tag{21}$$

where the subscript k denotes the kth random variable; n_v is the number of random variables, and:

$$\mathbf{F}'_{\mathbf{k}} = \int\int\int_{\Omega}\left[a'_k\left(\frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial x}\frac{\partial \mathbf{N}}{\partial x}+\frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial y}\frac{\partial \mathbf{N}}{\partial y}+\frac{\partial \mathbf{N}^{\mathbf{T}}}{\partial z}\frac{\partial \mathbf{N}}{\partial z}\right)\bar{\mathbf{T}}-\mathbf{N}^{\mathbf{T}}\frac{\partial \theta'_k}{\partial t}\right]\mathrm{d}\Omega$$
$$+\int\int_{\mathrm{c}} a'_k\frac{\bar{b}}{\bar{\lambda}}\mathbf{N}^{\mathbf{T}}\mathbf{N}\bar{\mathbf{T}}\,\mathrm{d}s+\int\int_{\mathrm{c}}\bar{a}\frac{b'_k\bar{\lambda}-\bar{b}\lambda'_k}{\bar{\lambda}^2}\mathbf{N}^{\mathbf{T}}\mathbf{N}\bar{\mathbf{T}}\,\mathrm{d}s-\int\int_{\mathrm{c}}\bar{a}\mathbf{N}^{\mathbf{T}}\left(\frac{b'_k\bar{\lambda}-\bar{b}\lambda'_k}{\bar{\lambda}^2}\bar{\mathbf{T}}_{\mathbf{a}}+\frac{\bar{b}}{\bar{\lambda}}\mathbf{T}'_{\mathbf{ak}}\right)\mathrm{d}s$$
$$-\int\int_{\mathrm{c}} a'_k\mathbf{N}^{\mathbf{T}}\frac{\bar{b}}{\bar{\lambda}}\bar{\mathbf{T}}_{\mathbf{a}}\,\mathrm{d}s\quad(k=1,2,\ldots,n_{\mathrm{v}}) \tag{22}$$

It should be noticed that the differentiation with respect to time t is involved in the finite element formulas of Eqs. (18) and (21). In practical calculation, the formulas should be written into following equations, which are based on the finite difference method with respect to time:

$$\left[\bar{\mathbf{K}}_{\mathbf{1}}+\frac{2}{\Delta\tau}\bar{\mathbf{K}}_{\mathbf{2}}\right]\bar{\mathbf{T}}_{\tau}+\left[\bar{\mathbf{K}}_{\mathbf{1}}-\frac{2}{\Delta\tau}\bar{\mathbf{K}}_{\mathbf{2}}\right]\bar{\mathbf{T}}_{\tau-\Delta\tau}+\bar{\mathbf{F}}_{\tau-\Delta\tau}+\bar{\mathbf{F}}_{\tau}=\mathbf{0} \tag{23}$$

$$\left[\bar{\mathbf{K}}_{\mathbf{1}}+\frac{2}{\Delta\tau}\bar{\mathbf{K}}_{\mathbf{2}}\right]\mathbf{T}'_{\mathbf{k}\tau}+\left[\bar{\mathbf{K}}_{\mathbf{1}}-\frac{2}{\Delta\tau}\bar{\mathbf{K}}_{\mathbf{2}}\right]\mathbf{T}'_{\mathbf{k}(\tau-\Delta\tau)}+\mathbf{F}'_{\mathbf{k}(\tau-\Delta\tau)}+\mathbf{F}'_{\mathbf{k}\tau}=\mathbf{0} \tag{24}$$

in which $\Delta\tau$ is time interval.

From Eq. (17), the expectation of temperature $\bar{\mathbf{T}}$ can be obtained. Moreover, from Eq. (21), $\mathbf{T}'_{\mathbf{k}}$ can be calculated. As a result, the covariance of temperature field can be obtained as follows:

$$\mathrm{cov}({}^{(i)}T,{}^{(j)}T)=\sum_{k=1}^{n_{\mathrm{v}}}\sum_{l=1}^{n_{\mathrm{v}}}{}^{(i)}T'_k\big|_{\mathbf{X}=\bar{\mathbf{X}}}{}^{(j)}T'_l\big|_{\mathbf{X}=\bar{\mathbf{X}}}\mathrm{cov}(X_k,X_l)\quad(i,j=1,2,\ldots,N_p) \tag{25}$$

in which ${}^{(i)}T$ denotes the temperature at the ith node, and ${}^{(i)}T'_k$ denotes the differentiation of the temperature at the ith node with respect to the kth random variable.

5. Influences of random processes

It is clear that the aforementioned studies have not considered time effect of the random hydration of concrete and random environmental temperature, but only the influences of random material parameters. If, however, the influences of both random hydration heat and random environmental temperature are taken into account in the same equation (Eq. (1)), the problem will become too difficult to be solved. Considering that the thermal problem discussed is within the scope of linear system. Therefore, the heat conduction equation (Eq. (1)) can be separated into two equations: (a) heat conduction equation-I for environmental random temperature process and (b) heat conduction equation-II for random hydration heat process.

(a) Heat conduction equation-I

$$\frac{\partial T_1}{\partial t}=a\nabla^2T_1+\frac{\partial\bar{\theta}}{\partial t},\quad T_1|_{t=0}=T_0(x,y,z),\qquad \frac{\partial T_1}{\partial n}\bigg|_{\mathrm{c}}+\frac{b}{\lambda}T_1\bigg|_{\mathrm{c}}=\frac{b}{\lambda}T_a \tag{26}$$

According to Eq. (3), the above equation can be further separated into two parts as:

(a0) Heat conduction equation-I$^{(0)}$

$$\frac{\partial T_1^{(0)}}{\partial t}=a\nabla^2T_1^{(0)}+\frac{\partial\bar{\theta}}{\partial t},\quad T_1^{(0)}\Big|_{t=0}=T_0(x,y,z),\quad \frac{\partial T_1^{(0)}}{\partial n}\bigg|_{\mathrm{c}}+\frac{b}{\lambda}T_1^{(0)}\bigg|_{\mathrm{c}}=\frac{b}{\lambda}T_m \tag{27}$$

and (a1) Heat conduction equation-I$^{(\mathrm{i})}$ $(i=1,2,\ldots,n_a)$

$$\frac{\partial T_1^{(i)}}{\partial t} = a\nabla^2 T_1^{(i)} + \frac{\partial \bar{\theta}}{\partial t}, \quad T_1^{(i)}\Big|_{t=0} = 0, \quad \frac{\partial T_1^{(i)}}{\partial n}\bigg|_c + \frac{b}{\lambda} T_1^{(i)}\bigg|_c = \frac{b}{\lambda} T_{ai} \tag{28}$$

(b) Heat conduction equation-II

$$\frac{\partial T_2}{\partial t} = a\nabla^2 T_2 + \frac{\partial \Delta\theta}{\partial t}, \quad T_2|_{t=0} = 0, \quad \frac{\partial T_2}{\partial n}\bigg|_c + \frac{b}{\lambda} T_2\bigg|_c = 0 \tag{29}$$

Obviously, T_2 is produced by the fluctuation part of hydration heat and hence $\bar{T}_2 = 0$. From Eqs. (26)–(29), we have:

$$T = T_1 + T_2 = T_1^{(0)} + \sum_{i=1}^{n_a} T_1^{(i)} + T_2 \tag{30}$$

which satisfies Eq. (1).

Upon calculating the random temperature of each part from the above heat conduction equations, the total random temperature field can be obtained:

$$\bar{T} = \bar{T}_1 + \bar{T}_2 = \bar{T}_1^{(0)} + \sum_{i=1}^{n_a} \bar{T}_1^{(i)} + \bar{T}_2 \tag{31}$$

$$\mathrm{var}(T) = \mathrm{var}(T_1) + \mathrm{var}(T_2) = \mathrm{var}(T_1^{(0)}) + \sum_{i=1}^{n_a} \mathrm{var}(T_1^{(i)}) + \mathrm{var}(T_2) \tag{32}$$

in which covariance among the three parts is neglected. Actually, after the differentiation of temperature with respect to random variables have been calculated, the covariance among the three parts can also be obtained, and we have:

$$\mathrm{var}(T) = \mathrm{var}(T_1) + \mathrm{var}(T_2) + 2\,\mathrm{cov}(T_1, T_2) \tag{33}$$

5.1. Numerical procedure for heat conduction equation-$I^{(0)}$

In this case, as random process factors are not involved (see Eq. (27)), the method of instantaneous variational principle and corresponding stochastic finite element formulas can be used. However, it should be mentioned that the effect of random initial temperature T_0 is not explicitly taken into account by the aforementioned methods. The initial temperature can be written as

$$\begin{aligned} &T_0 = \bar{T}_0 + \Delta T_0 \\ &\mathrm{var}(T_{T_0}) = \mathrm{var}(T_{\bar{T}_0 + \Delta T_0}) = \mathrm{var}(T_{\Delta T_0}) \end{aligned} \tag{34}$$

in which, the symbol T_{T_0} denotes temperature field caused only by the initial temperature field T_0.

Notice that in Eqs. (23) and (24), where only the effect of initial temperature is taken into account, we have $\mathbf{F} = \mathbf{0}$ and $\mathbf{F}' = \mathbf{0}$. Therefore, a recursive finite element formula for $\mathrm{var}(T_{\Delta T_0})$ can be written as follows

$$\mathrm{var}(\mathbf{T_{T_0}})_i \approx -\left[\left(\bar{\mathbf{K}}_\mathbf{1} - \frac{2}{\Delta\tau_i}\bar{\mathbf{K}}_\mathbf{2}\right) \Big/ \left(\bar{\mathbf{K}}_\mathbf{1} + \frac{2}{\Delta\tau_i}\bar{\mathbf{K}}_\mathbf{2}\right)\right]^2 \mathrm{var}(\mathbf{T_{T_0}})_{i-1} \tag{35}$$

in which, $\mathrm{var}(\mathbf{T_{T_0}})_i$ denotes column matrix of temperature variance caused by the random part of $\mathbf{T_0}$ at the ith time step. $\Delta\tau_i = t_i - t_{i-1}$. Therefore, the influence of $\mathbf{T_0}$ becomes smaller and smaller with time increasing.

As the direct use of Eq. (34) to calculate the random effect of initial temperature is very time-consuming, the convenient method is that T_0 is still regarded as a basic random variable in all calculations of Eqs. (17)–(25), so long as letting $\partial T/\partial T_0 = 1$ when $t = 0$.

5.2. Stochastic spectral response finite element method

According to the theory of random vibration, for a time invariant linear system, under excitation of random process $A\exp[\mathrm{i}\omega(t-t_0)]$, structural response Y can be obtained by the spectral response method (Lin, 1967)

$$Y = Ah(\omega)\exp[\mathrm{i}\omega(t-t_0)] \tag{36}$$

in which $h(\omega)$ is the complex frequency response function. In the aforementioned heat conduction equation-$\mathbf{I}^{(\mathrm{i})}$ $(i = 1, 2, \ldots, n_a)$, the environmental temperature T_{ai} $(i = 1, 2, \ldots, n_a)$ is a periodic stationary random process. Therefore, the spectral response method can be used. In ordinary analysis, however, the frequency response function is usually considered as a definite function. In order to take the effect of random material parameters into account, this paper takes the complex frequency response function as a function of both random material parameters and the random variable A. Statistics of the random function can be obtained by stochastic spectral response finite element method.

As the environmental temperature T_{ai} is a spatial random process, after the discretization of finite elements, it should be written as

$$\mathbf{T_{ai}} = \mathbf{A}\exp[\mathrm{i}\omega(t-t_0)] \tag{37}$$

Hence, corresponding temperature column matrix can be written as

$$\mathbf{T} = \mathbf{h}(\omega)\mathbf{A}\exp[\mathrm{i}\omega(t-t_0)] = \mathbf{H}(\omega)\exp[\mathrm{i}\omega(t-t_0)] \tag{38}$$

in which $\mathbf{H}(\omega) = \mathbf{h}(\omega)\mathbf{A}$, while $\mathbf{h}(\omega)$ is complex frequency response function matrix. The ith column of $\mathbf{h}(\omega)$ is the complex frequency response function of all nodes caused by the excitation of the ith node. Substituting Eqs. (37) and (38) into finite element formulas of heat conduction equation: $\mathbf{K_1T} + \mathbf{K_2}\partial\mathbf{T}/\partial t + \mathbf{F} = \mathbf{0}$, we have

$$\mathbf{K_1H}(\omega)\exp[\mathrm{i}\omega(t-t_0)] + \mathrm{i}\omega\mathbf{K_2H}(\omega)\exp[\mathrm{i}\omega(t-t_0)] + \mathbf{F} = \mathbf{0} \tag{39}$$

$$\mathbf{F} = -\sum_e a\int\int_{\mathrm{c}_e}\frac{b}{\lambda}N_i\mathbf{N}\mathbf{A}\exp[\mathrm{i}\omega(t-t_0)]\,\mathrm{d}s \tag{40}$$

Eliminating $\exp[\mathrm{i}\omega(t-t_0)]$ from the both hands of Eq. (39), we obtain

$$(\mathbf{K_1} + \mathrm{i}\omega\mathbf{K_2})\mathbf{H}(\omega) = \sum_e a\int\int_{\mathrm{c}_e}\frac{b}{\lambda}N_i\mathbf{N}\mathbf{A}\,\mathrm{d}s \tag{41}$$

The above equation is finite element formula of spectral response method for random temperature field. Based upon perturbation method, the corresponding stochastic finite element formulas can be deduced as:

(a) The zero order equation

$$(\bar{\mathbf{K}}_\mathbf{1} + \mathrm{i}\omega\bar{\mathbf{K}}_\mathbf{2})\bar{\mathbf{H}}(\omega) = \sum_e \bar{a}\int\int_{\mathrm{c}_e}\frac{\bar{b}}{\bar{\lambda}}N_i\mathbf{N}\bar{\mathbf{A}}\,\mathrm{d}s \tag{42}$$

(b) The first order equation

$$(\bar{\mathbf{K}}_\mathbf{1} + \mathrm{i}\omega\bar{\mathbf{K}}_\mathbf{2})\mathbf{H}'_\mathbf{k}(\omega) = \mathbf{F}'_\mathbf{k} \quad (k = 1, 2, \ldots, n_\mathrm{v}) \tag{43}$$

$$\mathbf{F}'_{\mathbf{k}} = -(\mathbf{K}'_{\mathbf{1k}} + \mathrm{i}\omega\mathbf{K}'_{\mathbf{2k}})\bar{\mathbf{H}}(\omega) + \sum_e a'_k \int\int_{c_e} \frac{\bar{b}}{\bar{\lambda}} N_i \mathbf{N}\bar{\mathbf{A}}\,\mathrm{d}s + \sum_e \bar{a} \int\int_{c_e} \frac{b'_k\bar{\lambda} - \bar{b}\lambda'_k}{\bar{\lambda}^2} N_i \mathbf{N}\bar{\mathbf{A}}\,\mathrm{d}s$$
$$+ \sum_e \bar{a} \int\int_{c_e} \frac{\bar{b}}{\bar{\lambda}} N_i \mathbf{N}\mathbf{A}'_{\mathbf{k}}\,\mathrm{d}s \quad (k = 1, 2, \ldots, n_v) \tag{44}$$

From Eqs. (42)–(44), $\bar{\mathbf{H}}(\omega)$ and $\mathbf{H}'_{\mathbf{k}}(\omega)$ $(k = 1, 2, \ldots, n_v)$ can be obtained. According to the theory of complex random process (Papoulis, 1965), auto-correlation function matrix of $\mathbf{T}$ can be written as

$$\mathbf{R}(\omega, t_1, t_2, t_0) = \mathrm{E}[\mathbf{T}(\omega, t_1, t_0)\mathbf{T}^*(\omega, t_2, t_0)] \tag{45}$$

in which $\mathbf{T}^*(\omega, t_2, t_0)$ is conjugate of $\mathbf{T}(\omega, t_2, t_0)$. $\mathrm{E}[\cdot]$ denotes expectation. Substituting Eq. (38) into the above equation, we obtain

$$\mathbf{R}(\omega, t_1, t_2, t_0) = \mathrm{E}\left\{\mathbf{H}(\omega)\mathrm{e}^{\mathrm{i}\omega(t_1 - t_0)}\left[\mathbf{H}(\omega)\mathrm{e}^{\mathrm{i}\omega(t_2 - t_0)}\right]^{*\mathrm{T}}\right\} \tag{46}$$

Notice that

$$\mathbf{H}(\omega) \approx \bar{\mathbf{H}}(\omega) + \sum_{k=1}^{n_v} \mathbf{H}'_{\mathbf{k}}(\omega)\Delta X_k \tag{47}$$

which can be substituted into Eq. (46) as:

$$\mathbf{R}(\omega, t_1, t_2, t_0) \approx \bar{\mathbf{H}}(\omega)\,\mathrm{e}^{\mathrm{i}\omega(t_1 - t_0)}\left[\bar{\mathbf{H}}(\omega)\,\mathrm{e}^{\mathrm{i}\omega(t_2 - t_0)}\right]^{*\mathrm{T}}$$
$$+ \sum_{k=1}^{n_v}\sum_{l=1}^{n_v} \mathbf{H}'_{\mathbf{k}}(\omega)\,\mathrm{e}^{\mathrm{i}\omega(t_1 - t_0)}\left[\mathbf{H}'_{\mathbf{k}}(\omega)\,\mathrm{e}^{\mathrm{i}\omega(t_2 - t_0)}\right]^{*\mathrm{T}} \mathrm{cov}(X_k, X_l) \tag{48}$$

When $t_1 = t_2$, the above equation becomes

$$\mathbf{R}(\omega, t, t_0) \approx \bar{\mathbf{H}}(\omega)\left[\bar{\mathbf{H}}(\omega)\right]^{*\mathrm{T}} + \sum_{k=1}^{n_v}\sum_{l=1}^{n_v} \mathbf{H}'_{\mathbf{k}}(\omega)\left[\mathbf{H}'_{\mathbf{k}}(\omega)\right]^{*\mathrm{T}} \mathrm{cov}(X_k, X_l) \tag{49}$$

Obviously, diagonal column of $\mathbf{R}(\omega, t, t_0)$ is variance-mean square of $\mathbf{T}$

$$\mathrm{E}(\mathbf{T}^2) = \left|\bar{\mathbf{H}}(\omega)\right|^2 + \sum_{k=1}^{n_v} \left|\mathbf{H}'_{\mathbf{k}}(\omega)\right|^2 \mathrm{var}(X_k) \tag{50}$$

Therefore, the variance of temperature field $\mathbf{T} = \left({}^{(1)}T, {}^{(2)}T, \ldots, {}^{(N_p)}T\right)^{\mathrm{T}}$ can be written as

$$\mathrm{var}\left({}^{(i)}T\right) = \mathrm{E}\left({}^{(i)}T^2\right) - \mathrm{E}\left({}^{(i)}T\right)\mathrm{E}\left({}^{(i)}T^*\right) = \sum_{k=1}^{n_v} \left|{}^{(i)}H'_k(\omega)\right|^2 \mathrm{var}(X_k) \quad \left(i = 1, 2, \ldots, N_p\right) \tag{51}$$

5.3. Numerical procedure for the heat conduction equation-II

It is known from Eq. (29) that only two random factors θ_0 and m_1 are involved in the heat conduction equation-II. Since the non-stationary random process $\partial\theta/\partial t$ has been simplified as Eq. (8) and $\bar{\mathbf{T}}_{\mathbf{2}} = \mathbf{0}$, only the first order finite element equation is needed, which can be written as:

$$\bar{\mathbf{K}}_{\mathbf{1}}\bar{\mathbf{T}}'_{\mathbf{2k}} + \bar{\mathbf{K}}_{\mathbf{2}}\frac{\partial \mathbf{T}'_{\mathbf{2k}}}{\partial t} + \mathbf{F}'_{\mathbf{2k}} = 0 \quad (k = 1, 2) \tag{52}$$

$$\mathbf{F}'_{\mathbf{2k}} = -\int\int_{\Omega} \mathbf{N}^{\mathbf{T}}\left[\bar{m}_1 e^{-\bar{m}_1 t}\Delta\theta'_{0k} + (\bar{\theta}_0 - \bar{\theta}_0\bar{m}_1 t e^{-\bar{m}_1 t})\Delta m'_{1k}\right] dx\,dy\,dz \quad (k = 1, 2) \tag{53}$$

Upon calculating $\mathbf{T}'_{\mathbf{2k}}$ from the above equations, the auto-covariance of the temperature field $\mathbf{T_2}$ can be obtained as follows:

$$\mathrm{cov}\left[{}^{(i)}T_2, {}^{(j)}T_2\right] = \sum_{k=1}^{2}\sum_{l=1}^{2} {}^{(i)}T'_2\big|_{\mathbf{X}=\bar{\mathbf{X}}}\,{}^{(j)}T'_2\big|_{\mathbf{X}=\bar{\mathbf{X}}}\mathrm{cov}(X_k, X_l) \tag{54}$$

in which $X_1 = \Delta\theta_0$, $X_2 = \Delta m_1$.

It should be mentioned that, although this heat conduction problem is involved with only two random factors, when the random field properties of which should be taken into account, more than two random variables will appear after the random fields have been discretized by the local average method. Therefore, in Eqs. (52)–(54), when random field properties should be taken into account, only when the correlation functions of the random fields are equal to 1.0, the number of random variables n_v equals 2; otherwise, $n_v > 2$, and the subscript k should be from 1 to n_v ($n_v > 2$).

6. Numerical applications

According to the methods suggested in the above sections, a program named STEM (in FORTRAN Language) is coded. This program integrates various kinds of random factors such as random initial temperature, random material parameters and random environmental etc. to make easier the calculation of random temperature fields of large mass concrete structures during both construction period and operation period. Numerical examples are followed to testify the program and the presented methods.

6.1. Example 1

Considering an unconstrained concrete slab shown in Fig. 1 (Wang, 1985), air temperature on the right side of the slab is $T_{a1} = 0$, and surficial heat emission coefficient is $b_1 = 6.0$ kcal/m^2 h °C. Air temperature on the left side is $T_{a2} = A\cos\omega(t - t_0)$, and $b_2 = 12.0$ kcal/m^2 h °C. $\omega = 7.169 \times 10^{-4}$ 1/h. Thickness of the slab is 2.6 m. Coefficient of thermal conductivity is $\lambda = 1.25$ kcal/m h °C. Temperature transfer coefficient is $a = 0.00217$ m^2/h. Finite element mesh is shown in Fig. 2.

When only the fluctuation A is considered as random variable ($\bar{A} = 8$°C, standard deviation is 1.6°C, Gaussian distribution), the result is calculated as in Table 1. This result is obtained by considering the complex frequency response function as a function of random variable A (see Eq. (50)). The theoretical solution was given by Wang (1985). The same result can be obtained by the spectral finite element method suggested by Bazant and Wang (1984a). It is known that the satisfied precision comes from the spectral approach, because the material parameters are constants.

It can be seen from Table 1 that, although the standard deviation of air temperature is 1.6°C at the boundary of $x = 2.6$ m, the corresponding standard deviation is less than 1.6°C. Further, the air temperature is a constant (0.0°C) at the boundary of $x = 0.0$ m, while the corresponding standard deviation is not zero because of the heat conduction.

6.2. Example 2

A, a, b_1, b_2 and λ in the above example are further studied as random variables, all supposed to be Gaussian distributed and variance coefficients are 0.2. At this stage, both random environmental

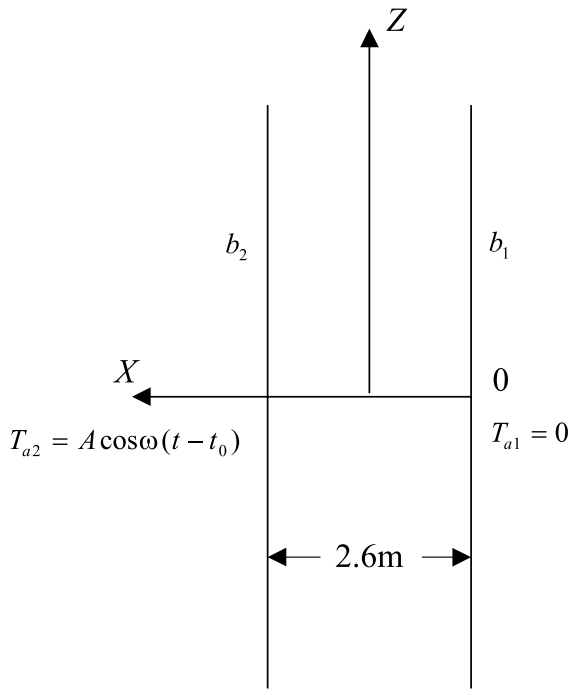

Fig. 1. Unconstrained concrete slab.

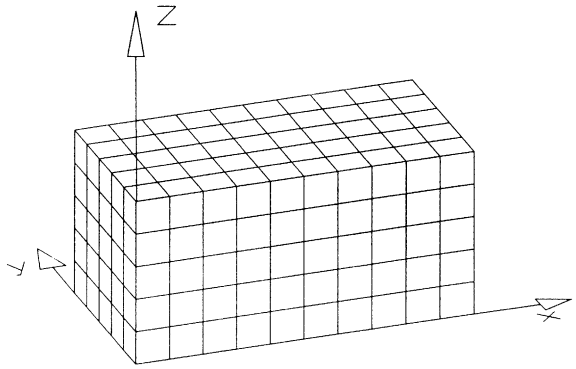

Fig. 2. Finite element mesh.

Table 1
Standard deviation (only A is random variable)

x (m)	0.0	1.3	2.6
Theoretical	0.0800	0.8000	1.5200
This paper	0.0809	0.8001	1.5212

Table 2
Standard deviation (A, a, b_1, b_2 and λ are random variables)

x (m)	0.0	1.3	2.6
SFEM	0.1993	0.9934	1.5396
Monte-Carlo	0.1974	0.9914	1.5389

temperature (random excitation) and random material parameters are involved. However, theoretical solution can not be obtained, and the spectral finite element method is not applicable to this problem either. Therefore, Monte-Carlo simulation is used to verify the results calculated by the suggested stochastic finite element method. The results are shown in Table 2.

It can be seen from Table 2 that, although the random properties of material parameters are taken into account, the standard deviation of the temperature at the left side of the concrete slab is still smaller than that of the air temperature. This means standard deviation of structural temperature at the boundary (strictly speaking is near the boundary) is mainly controlled by statistics of the environmental temperature.

6.3. Example 3

In the previous examples, the hydration heat of concrete was not involved. Suppose that the parameters of hydration heat θ_0 and m_1 are all Gaussian distributed and the variance coefficients are all 0.2. The expectation of the temperature at the point $x = 1.3$ m is shown in Fig. 3. Corresponding standard deviation is given in Fig. 4. Monte-Carlo simulation results are listed both in Figs. 3 and 4. Auto-correlation coefficient function on time axis is shown in Fig. 5. As the sizes of slab along y-axis and z-axis are infinitive, the auto-correlation coefficient functions along y-axis and z-axis are equal to 1.0. Auto-correlation coefficient function along x-axis is shown in Fig. 6.

From Figs. 3 and 4, it can be seen that at the beginning period of construction the standard deviation of temperature, because of the hydration heat of concrete, increases together with the expectation of temperature. However, at the later period (after about seven days), the influence of hydration heat diminishes and vanishes after about 90 days. The expectation of temperature gradually tends to be that of the case

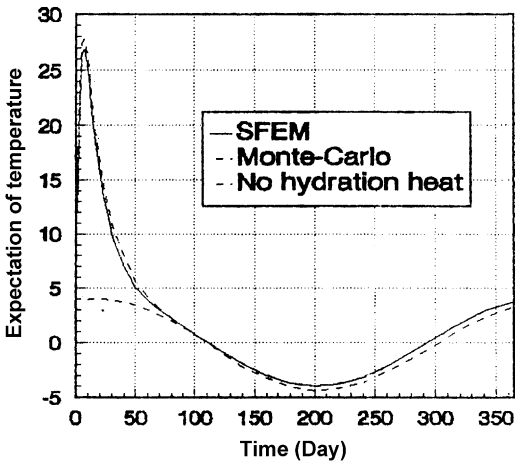

Fig. 3. Expectation of temperature ($x = 1.3$ m).

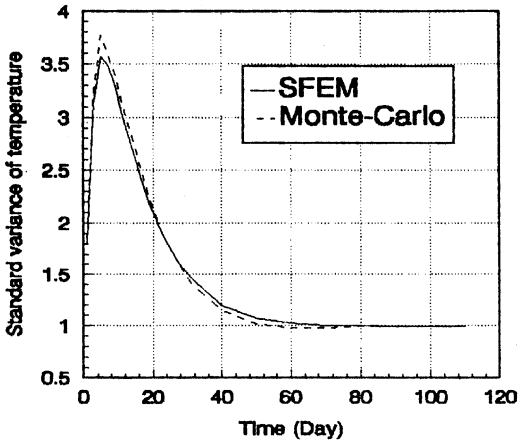

Fig. 4. Standard deviation of temperature ($x = 1.3$ m).

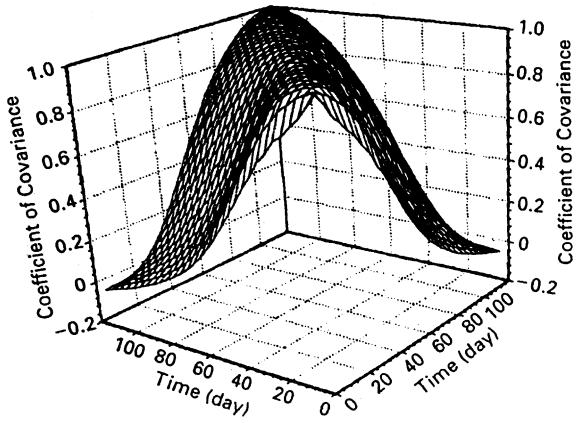

Fig. 5. Auto-correlation function on time axis.

without hydration heat (see Fig. 3). In Fig. 3, the cosine curve is caused only by the environmental temperature. After about 90 days, the temperature becomes the same with that of the case without hydration heat. Meanwhile, as a periodic random process can be regarded as a stationary random process, the standard deviation, hence, should be constant, the standard deviation of the temperature of the concrete slab at $x = 1.3$ m becomes a constant (about 0.9934, the same with the value in Table 2) after about 90 days.

It can be concluded from this example that for the specific problem the auto-correlation of the temperature on time axis will become zero when the time interval is bigger than about 120 days. This indicates that when thermal stresses need be calculated, the temperature of mass concrete structures should be regarded as a random process. Moreover, as the size of the concrete slab along the x-axis is negligible in comparison with that of y-axis and z-axis, the random temperature is strongly auto-correlated along the x-axis except that of the boundary.

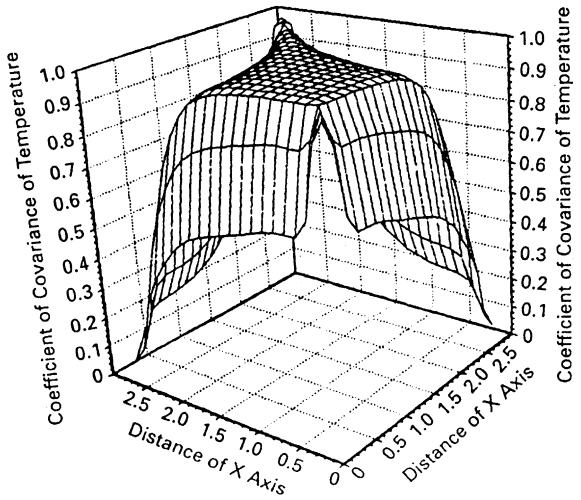

Fig. 6. Auto-correlation function on x-axis.

7. Conclusions

Owing to the various kinds of random factors, it is usually very difficult and complicated to calculate random temperature field of mass concrete structures. When only the random property of environmental temperature needs to be taken into account, the spectral finite element method suggested by Bazant and Wang is a very efficient method. Although for this simple case the suggested stochastic finite element method can also be implemented, it appears that it is not necessary to use this advanced tool. It is important that the difference between the spectral finite element method and the suggested stochastic finite element, in this case, is that the complex frequency response function used by the spectral finite element is a definite function. In the suggested stochastic finite element method, the complex frequency response function is a function of random fluctuation of the environmental temperature.

When random properties of material parameters should be further studied, the spectral finite element method can not be used. By considering the complex frequency response function as a function of both random fluctuation of environmental temperature and random material parameters, the suggested method (the stochastic spectral response finite element method) is a useful method for this problem. Moreover, as the hydration heat of concrete playing a very important role in the temperature field of concrete, should be regarded as a non-stationary process, it is formidable to calculate its effect when the non-stationary process is directly dealt with. In this paper, the non-stationary process is simplified by using the first order Taylor expansion series, and therefore the corresponding stochastic finite element formulas can be obtained.

In short, the suggested methods are useful for the calculation of random temperature field of mass concrete structures and thermal control of practical engineering where random variability should be taken into account.

Besides, some more conclusions can be obtained:

1. During the beginning period of construction, the hydration heat of concrete plays a dominant part for the temperature filed of concrete structures. The standard deviation increases synchronously with the increasing of the expectation of temperature.
2. During the later period (after about seven days for the specific example), the influences of the hydration heat of concrete becomes weaker and weaker. The temperature tends to be a periodic one caused only by

the environmental temperature. The standard deviation of the temperature gradually becomes a constant.

3. At the location near boundary, the temperature of concrete is largely dependent upon the environmental temperature during both construction period and operation period. However, owing to the heat conduction and random material parameters, the standard deviation of the temperature is not zero even at the boundary, where the environmental temperature is a constant. Moreover, for the specific example, the standard deviation of the temperature of concrete at the boundary is smaller than the standard deviation of environmental temperature although the variability of both material parameters and environmental temperature is taken into account.
4. When thermal stresses need to be calculated, the temperature of the concrete should be regarded as a random process on time axis. It is also concluded that when the thickness is negligible, compared with the size of other two directions, the auto-correlation function along the direction of thickness is about 1.0 except for that of the boundary. In practical engineering of mass concrete structures such as concrete dams, the concrete structures are usually constructed layer by layer (especially in the case of RCC or RCD dams). Therefore, when calculating the thermal stress, the random temperature field of concrete within a layer can be regarded as a random variable along the vertical direction.

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Appendix A. Local average method for 3-D vector random field

In a local coordinate system $o'_i x'_i y'_i z'_i$, it is supposed that in the $o'_i x'_i y'_i$ plane, the correlation function of the random field is a 2-D function $\rho_{12}^{(i)}(\Delta x'_i, \Delta y'_i)$ without varying with the value of z'_i. While, along the z'_i axis, the correlation function is a 1-D function $\rho_3^{(i)}(\Delta z'_i)$ without varying with the value of x'_i, y'_i. This random field is thus called 3-D partly separable random field. Moreover, if $\rho_{12}^{(i)}(\Delta x'_i, \Delta y'_i)$ is also separable, the random field becomes a perfectly separable random field. In the global space, i.e. in the global coordinate system, there may be several such kind of random fields. The 3-D vector random field is supposed to consist of these local (scalar) random fields.

It can be seen that there are four advantages in using these random fields to describe a practical engineering. Firstly, the 3-D properties of actual random field can be approximately reflected. Secondly, it is not very difficult to simulate the random field in terms of numerical discretization in conjunction with SFEM. Thirdly, it is very easy to reduce the random field model to that of 2-D or 1-D cases. For instance, if $\rho_3^{(i)}(\Delta z'_i)$ is set to be 1, the ith local random field becomes 2-D random fields. Or if $\rho_{12}^{(i)}(\Delta x'_i, \Delta y'_i) = 1$, the random field becomes 1-D. Finally, every local (scalar) random field is defined in its own local coordinate system and these local coordinate systems form a mixed coordinate system, which, however, has the same global coordinate system. Therefore, it is convenient to choose one or more local random fields to simulate the realistic random fields, if necessary.

Suppose the vector random field $\{S(x,y,z)\}$ consists of n local scalar random field: $S_i(x'_1, y'_1, z'_1)$, $S_2(x'_2, y'_2, z'_2), \ldots, S_n(x'_n, y'_n, z'_n)$. In the global coordinate system, the correlation structure can be expressed as a matrix of correlation function $[\rho_{ij}(\Delta x, \Delta y, \Delta z)]$ $(i,j = 1,2,\ldots,n)$, in which, when $i = j$, we have

$$\rho_{ij}(\Delta x, \Delta y, \Delta z) = \rho^{(i)}(\Delta x'_i, \Delta y'_i, \Delta z'_i) \quad (i = j = 1,2,\ldots,n) \tag{A.1}$$

in which, $\rho^{(i)}(\cdot)$ is correlation function of the ith random field $S_i(x'_i, y'_i, z'_i)$ in the local coordinate system $o'_i x'_i y'_i z'_i$. If the orientation cosine of the ith local coordinate system is $(l_{ix}, m_{ix}, n_{ix})(x,y,z)$, we have

$$(\Delta x_i', \Delta y_i', \Delta z_i')^{\mathrm{T}} = \hat{\mathbf{T}}^{(i)}(\Delta x, \Delta y, \Delta z)^{\mathrm{T}}$$

$$\hat{\mathbf{T}}^{(i)} = \begin{bmatrix} \hat{\mathbf{T}}_{li}^{\mathrm{T}} \\ \hat{\mathbf{T}}_{mi}^{\mathrm{T}} \\ \hat{\mathbf{T}}_{ni}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} l_{ix} & l_{iy} & l_{iz} \\ m_{ix} & m_{iy} & m_{iz} \\ n_{ix} & n_{iy} & n_{iz} \end{bmatrix} \tag{A.2}$$

Therefore

$$\rho_{ij}(\Delta x, \Delta y, \Delta z) = \rho^{(i)}(\hat{\mathbf{T}}_{li}^{\mathrm{T}}\mathbf{D}_{\mathrm{w}}, \hat{\mathbf{T}}_{mi}^{\mathrm{T}}\mathbf{D}_{\mathrm{w}}, \hat{\mathbf{T}}_{ni}^{\mathrm{T}}\mathbf{D}_{\mathrm{w}}) \quad (i = j = 1, 2, \ldots, n) \tag{A.3}$$

in which $\mathbf{D}_{\mathrm{w}} = (\Delta x, \Delta y, \Delta z)^{\mathrm{T}}$.

For a scalar random field $S(x, y, z)$, the local average in the element V_i can be written as follows:

$$S_{Vi} = \frac{1}{V_i}\int\int\int_{Vi} S(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{A.4}$$

If the element V_i is a 3-D iso-parametric element, the above equation can be expressed as

$$\begin{aligned} S_{Vi} &= \frac{1}{V_i}\sum_{k=1}^{n_{\mathrm{g}}} S(\xi_k, \eta_k, \zeta_k)|J|_k H_k \\ V_i &= \sum_{k=1}^{n_{\mathrm{g}}} H_k|J|_k \end{aligned} \tag{A.5}$$

in which $S(\xi_k, \eta_k, \zeta_k)$ is the value of $S(x, y, z)$ at the kth Gaussian point within the ith element; ξ_k, η_k, ζ_k are the local coordinates of the kth Gaussian point; H_k is the Gaussian weighted coefficient; n_{g} is the number of Gaussian points; $|J|$ is the Jacobi determinant. If the random field $S(x, y, z)$ is homogeneous with mean value m and standard deviation σ, from Eq. (A.6), it is obtained that

$$\begin{aligned} \mathrm{E}[S_{Vi}] &= \frac{1}{V_i}\sum_{k}^{n_{\mathrm{g}}} \mathrm{E}[S(\xi_k, \eta_k, \zeta_k)]H_k|J|_k = m \\ \mathrm{cov}\big[S_{Vi}, S_{vj}\big] &= \mathrm{E}\big[(S_{Vi} - E(S_{Vi}))\big(S_{Vj} - E(S_{Vj})\big)\big] \\ &\approx \mathrm{E}\left[\frac{1}{V_i V_j}\sum_{k=1}^{n_{\mathrm{g}}}\sum_{k=1}^{n_{\mathrm{g}}} H_k|J|_k H_l|J|_l (S_{Vik} - \mathrm{E}(S_{Vik}))\big(S_{Vjl} - \mathrm{E}(S_{Vjl})\big)\right] \\ &= \sum_{k=1}^{n_{\mathrm{g}}}\sum_{l=1}^{n_{\mathrm{g}}} w_{ik}w_{jl}\mathrm{cov}\big[S_{Vik}, S_{Vjl}\big] \end{aligned} \tag{A.6}$$

in which $S_{Vik} \equiv S_{Vi}(\xi_k, \eta_k, \zeta_k)$, $S_{Vjl} \equiv S_{Vj}(\xi_l, \eta_l, \zeta_l)$, and

$$\begin{aligned} &w_{ik} = H_k|J|_k V_i, \qquad w_{jl} = H_l|J|_l/V_j \\ &\mathrm{cov}\big(S_{Vik}, S_{Vjl}\big) = \sigma^2\rho\big(\Delta x_{ik,jl}, \Delta y_{ik,jl}, \Delta z_{ik,jl}\big) \end{aligned} \tag{A.7}$$

where $\mathrm{cov}\big(S_{Vik}, S_{Vjl}\big)$ is the covariance between the kth Gaussian point in the ith element and the lth Gaussian point in the jth element; $\rho\big(\Delta x_{ik,jl}, \Delta y_{ik,jl}, \Delta z_{ik,jl}\big)$ is the corresponding correlation function

$$\rho\big(\Delta x_{ik,jl}, \Delta y_{ik,jl}, \Delta z_{ik,jl}\big) = \rho_{12}\Big(\hat{\mathbf{T}}_l^{\mathrm{T}}\mathbf{D}_{ik,jl}, \hat{\mathbf{T}}_m^{\mathrm{T}}\mathbf{D}_{ik,jl}\Big)\rho_3\Big(\hat{\mathbf{T}}_n^{\mathrm{T}}\mathbf{D}_{ik,jl}\Big) \tag{A.8}$$

in which $\Delta x_{ik,jl}(x, y, z)$ is the projective distance between the two Gaussian points on the x (x, y, z) axis, which can be written in the finite element format

$$\Delta x_{ik,jl} = \sum_{m=1}^{n_g} \left[(N_m)_l x_{jm} - (N_p)_k x_{im} \right], \quad (x, y, z) \tag{A.9}$$

where $(N_m)_l$ and $(N_m)_k$ are the value of shape function at the lth and kth Gaussian point with respect to the mth node, respectively; x_{jm} and x_{im} are x coordinates at the mth node of the jth and the ith elements, respectively.

As for the 3-D vector random field, the auto-correlation function of the local average of each component random field can be obtained from the above equations, while the cross-correlation function between the local averages of different component random fields can be described as follows

$$\operatorname{cov}\left(S_{Vi}^{(p)}, S_{ji}^{(q)}\right) \approx \sum_{k=1}^{n_g} \sum_{l=1}^{n_g} w_{ik} w_{jl} \operatorname{cov}\left[S_{Vik}^{(p)}, S_{Vjl}^{(q)}\right], \quad (i \neq j) \tag{A.10}$$

in which

$$\operatorname{cov}\left[S_{Vik}^{(p)}, S_{Vjl}^{(q)}\right] = \sigma_p \sigma_q \rho_{ij}\left(\Delta x_{ik,jl}, \Delta y_{ik,jl}, \Delta z_{ik,jl}\right) \tag{A.11}$$

where $S_{Vik}^{(p)}, S_{Vjl}^{(q)}$ are the local averages at the kth and the lth Gaussian point of the pth and the qth scalar random fields in the ith and jth elements, respectively. σ_p and σ_q are the standard deviation of the pth and the qth homogeneous random field, respectively.

From the above descriptions, it is clear that after the finite element meshes and the auto-correlation or cross-correlation functions of scalar random fields have been obtained, the overall random fields can be approximately represented by a number of correlated random variables.

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